M-math 2nd year Final Exam
Subject : Fourier Analysis

Time : 2.30 hours Max.Marks 50.

1. Suppose that $\sum_{n\in\mathbb{Z}} |nC_n| < \infty$, where $C_n, n \in \mathbb{Z}$ are complex numbers. Show that if

$$f(\theta) := \sum_{n\in\mathbb{Z}} C_n e^{in\theta}$$

then

$$f'(\theta) = \sum_{n\in\mathbb{Z}} n\hat{f}(n)e^{in\theta}$$

where $\hat{f}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{in\theta}\, d\theta$. (10)

2. Let $\{P_r(\theta), 0 \le r < 1\}$ be the Poisson kernel, defined as

$$P_r(\theta) := \frac{1-r^2}{1+r^2-2rcos\theta}.$$

Show that for each $0 \le r < 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\, d\theta = 1 \quad \text{and} \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \{P_r(\theta)\}^2\, d\theta = \frac{1+r^2}{1-r^2}.$$

(10)

3. For $t > 0, x \in \mathbb{R}^d$, let $H(t,x) := \int\limits_{\mathbb{R}^d} e^{-4\pi^2 t|\xi|^2} e^{2\pi\xi\cdot x}\, d\xi$. Then show that

$$\partial_t H(t,x) = \sum_{j=1}^{d} \partial_j^2 H(t,x),\ t > 0, x \in \mathbb{R}^d,$$

where ∂_t, ∂_j are the partial derivatives with respect to t and x_j respectively. (10)

4. Show that the kernel $k_t(x) := \frac{t}{\pi(t^2+x^2)}, t > 0, x \in \mathbb{R}$ is an approximate identity. (10)

5. Let $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be the extension to $L^2(\mathbb{R}^d)$ of the Fourier transform on $L^1(\mathbb{R}^d)$ given by the Plancherel theorem. For $f \in L^2(\mathbb{R}^d)$, $a, b \in \mathbb{R}^d$ compute $\mathcal{F}(f_a)$ and $\mathcal{F}(f^b)$ where $f_a(x) := f(x-a)$ and $f^b(x) := e^{-2\pi i b\cdot x} f(x)$. (10)